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Casimir energies for spherically symmetric cavities

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Abstract

A general calculation of Casimir energies—in an arbitrary number of dimensions—for massless quantized fields in spherically symmetric cavities is carried out. All the most common situations, including scalar and spinor fields, the electromagnetic field and various boundary conditions are treated with the uppermost accuracy. The final results are given as analytical, closed expressions in terms of Barnes zeta functions. A direct numerical evaluation of the formulae is then performed, which yields highly accurate numbers of, in principle, arbitrarily good precision.

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1. Introduction

Calculations of Casimir energies in spherically symmetric situations have attracted the interest of physicists for well over 30 years now. Since the calculation of Boyer [1], who computed the Casimir energy for a conducting spherical shell and found, to his surprise, a repulsive force, many different situations in the spherically symmetric context have been considered. For example, dielectrics were included [2] (for the case of plane, parallel surfaces see [3]) and used later for possible explanations of sonoluminescence [4–8]. Moreover, enormous interest has been attracted by the MIT bag model in QCD [9–20] and, also, the influence of different boundary conditions has been considered in detail [21–23].

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Different methods have been used for dealing with the Casimir effect. Whereas in earlier times the Green function formalism was preferred, in recent years different approaches—which make use of contour integral representations of the involved spectral sums—are commonplace. Although the idea for this method, in the specific context of Casimir energies, goes back to the early days of the subject [24], a systematical, effective and simple application of this approach in various contexts has only recently been achieved [21–28].

The spectral sum which actually appears in the calculation depends on the regularization used and may include a cutoff function, to dampen high-frequency contributions [28] or, as in the zeta regularization technique [29], complex powers of the eigenvalues [21,22,26,30]. As a result, the details of the computation may differ slightly, for example in the specific integration contour chosen, but all of them share the elegance of this method.

In recent contributions we have further developed the zeta function technique, in combination with several contour integral representations. Given the deep connections among zeta functions, heat kernels and functional determinants [31–34], one advantage of the method is that it can be applied, alternatively, to the calculation of heat kernel coefficients [26] and functional determinants [27,30] (see also [35,36]), as well as Casimir energies [20–22]. This clearly shows that zeta functions serve as a unified framework in different areas of interest.

Here we want to pursue this idea, by using the zeta function framework in a precise analysis of the Casimir energy as a function of the dimension of space. Previously it had been shown that arbitrary space dimension can be treated elegantly by making use of Barnes zeta functions, where the dimension can be considered as a parameter [27, 37]. This has been applied to the calculation of heat-kernel coefficients and determinants and it will be here used to study the Casimir energy. Apart from dealing with arbitrary dimensions, we will introduce scalars, spinor fields and the electromagnetic field in a unified way, including the effects different sets of boundary conditions have on them. In spirit, our analysis is to be compared with that of Ambjørn and Wolfram in [38], with the difference that the role of the Epstein zeta function there is here played by the Barnes zeta function. For a recent analysis on the dimensional dependence of the Casimir energy for scalar fields with Dirichlet boundary conditions and the electromagnetic field in the presence of a spherical shell see [39, 40], where the space dimension D has been dealt with as a parameter, and results for (in principle) all values of real D have been obtained.

The paper is organized as follows. In the next section we briefly recall the definition of Casimir energies in terms of zeta functions. In section 3, we shortly describe the method and derive the formulae that are subsequently needed in the context of Casimir energy calculations [26, 27]. In section 4 we consider the case of a scalar field. For Dirichlet boundary conditions, the energy in dimensions $D = 2$ up to 9 is given there. The interior and the exterior regions are treated separately. Afterwards, the changes in the procedure needed for Robin boundary conditions are explained, and the corresponding formulae are derived. Given that the Casimir energy of the electromagnetic field is determined by using the Casimir energy of a scalar field satisfying Dirichlet boundary conditions (TE modes) and a scalar field satisfying Robin boundary conditions (TM modes), these forms constitute the basis for the electromagnetic case, and nothing else needs to be calculated, as will be later described in detail (section 6). Before that, section 5 is devoted to the spinor field. Local bag boundary conditions, as well as global spectral boundary conditions, are considered. In the concluding section 7, a summary of our main results, as well as details on how our method is indeed able to yield arbitrary accurate results, are given.

2. The Casimir energy

The Casimir energy of a quantum field $\Phi(t,\vec{x})$ inside a spherical shell is formally given by

$$E_{\mathrm{Cas}} = \frac{1}{2}\sum_k \omega_k \tag{2.1}$$

(we set $\hbar = c = 1$) with the one-particle energies $\omega_k = \sqrt{\lambda_k}$ being obtained from

$$-\Delta\phi_k(\vec{x}) = \lambda_k\phi_k(\vec{x}) \tag{2.2}$$

also fulfilling suitable boundary conditions. The field operator is, in our case, $A = \partial_t^2 - \Delta$, and we have $\Phi(t,\vec{x}) = \mathrm{e}^{-\mathrm{i}\omega t}\phi(\vec{x})$. The Laplacian Δ is that defined inside or outside the $D = (d+1)$-dimensional ball $B^D = \{\vec{x} \in \mathbb{R}^D \|\vec{x}\| \leqslant R\}$ and the fields $\phi(\vec{x})$ must satisfy appropriate boundary conditions at $\|\vec{x}\| = R$.

The Casimir energy as given by the formal expression (2.1) is ill defined and has to be regularized. In the ζ-function regularization procedure, one writes

$$E_{\mathrm{Cas}} = \frac{1}{2}\mu^{2s}\zeta(s-1/2)|_{s=0} \tag{2.3}$$

$$\zeta(s) = \sum_{\lambda_k\neq 0}\lambda_k^{-s}. \tag{2.4}$$

Here, μ is an arbitrary parameter with dimensions of mass to yield the correct dimension for all values of s, and $\zeta(s)$ is the ζ-function corresponding to the operator A. In some cases, E_{Cas} will be divergent and, as is known and will be seen later on, renormalization ambiguities may remain.

In order to calculate E_{Cas} according to the previous definition, we need information on the zeta function $\zeta(s)$ in a neighbourhood of $s = -1/2$. As we are dealing with operators in flat space, but satisfying boundary conditions on a d-dimensional sphere ($d = D-1$, the boundary of the D-dimensional ball), the eigenvalues will be implicitly given as the zeros of a polynomial $\tilde{P}(\tilde{Z}_\nu, \tilde{Z}'_\nu)$ involving Bessel or Hankel functions, according to whether one is considering the internal or the external domain, respectively. We will denote the associated zeta functions by $\zeta^{\mathrm{int}}(s)$ and $\zeta^{\mathrm{ext}}(s)$. The total Casimir energy will be the sum of the two terms, that is

$$E_{\mathrm{Cas}} = \frac{1}{2}\mu^{2s}\left[\zeta^{\mathrm{int}}(s-1/2) + \zeta^{\mathrm{ext}}(s-1/2)\right]\Big|_{s=0}. \tag{2.5}$$

With just a few modifications, which involve the phase of the zeta function (see [41] for precise details), all the considerations above can be extended to the Dirac operator. The basic construct turns out to be the zeta function of the square of the Dirac operator and one encounters a minus sign in equation (2.3).

3. The method

The method to be used here has been developed in the seminal papers [21, 26, 27] and permits us to compute the ζ-function starting from the (indirect) knowledge of the eigenvalues through an implicit relation of the kind

$$\tilde{P}(\tilde{Z}_{\nu_l}(\omega_{nl}R), \tilde{Z}'_{\nu_l}(\omega_{nl}R)) = 0 \qquad \lambda_{nl} = \omega_{nl}^2 \tag{3.1}$$

where $n, l \geqslant 0$ are the principal and azimuthal quantum numbers respectively. The degeneracy $d(l)$ of the eigenvalues and the index ν_l of the Bessel functions depend on l and on the dimension. Their explicit forms are strictly related to the fields and the boundary conditions.

The ζ-function can be expressed as an integral in the complex plane, that is

$$\zeta(s) = \sum_l \frac{d(l)}{2\pi\mathrm{i}}\int_\gamma k^{-2s}\frac{\partial}{\partial k}\ln k^{-b\nu_l}\tilde{P}(\tilde{Z}_{\nu_l}(k), \tilde{Z}'_{\nu_l}(k))\,\mathrm{d}k \qquad \mathrm{Re}\ s > \frac{D}{2} \tag{3.2}$$

where the open contour γ has to be chosen to run counterclockwise and to enclose all strictly positive solutions of equation (3.1). The additional factor $k^{-b\nu_l}$ has been inserted in order to cancel the pole at the origin, which is important when deforming the contour in the next step. In this way γ can also include the origin. Here b is a number which depends on the asymptotic behaviour of $\tilde{P}$ at the origin: in our cases it will be ± 1 for scalars, but for spin-$\frac{1}{2}$ with mixed boundary conditions it turns out to be ± 2.

For explicit calculation, it is convenient to write equation (3.2) as an integral on the real axis. This can be done by deforming the contour γ to the imaginary axis and by making the substitution $k \to \mathrm{i}y$. In general one has to be careful when deforming the contour that no poles in the plane Re $k \geqslant 0$ are hit. In fact, for Robin boundary conditions one has

$$\tilde{P}(\tilde{Z}_\nu, \tilde{Z}'_\nu) = \alpha \tilde{Z}_\nu(k) + k\tilde{Z}'_\nu(k) = (\alpha - \nu)\tilde{Z}_\nu(k) + k\tilde{Z}_{\nu-1}(k) = 0 \tag{3.3}$$

which may have solutions for $k \notin \mathbb{R}$ too, if $\alpha > \nu$. To avoid these cases, in the following we will consider $\alpha \leqslant \nu_0$ only, ν_0 corresponding to the smallest eigenvalue.

With this assumption we can write the ζ-function in the simpler form

$$\zeta(s) = \frac{\sin \pi s}{\pi} \sum_{l=0}^{\infty} d(l) \int_0^\infty y^{-2s} \frac{\partial}{\partial y} \ln[y^{-b\nu_l} P(\nu_l, y)] \, \mathrm{d}y \tag{3.4}$$

which is valid for $1/2 < \mathrm{Re}\ s < 1$ (for details see [26]). Here, $P(\nu, y) = P(Z_\nu(y), Z'_\nu(y))$ is a polynomial like $\tilde{P}$ (aside, possibly, from an irrelevant sign) and the $Z_\nu(y) = \tilde{Z}_\nu(\mathrm{i}y)$ are the modified Bessel functions corresponding to $\tilde{Z}$. In order to compute the Casimir energy we need the ζ-function at $s = -1/2$ and so we have to make an analytic continuation of equation (3.4).

With this aim, let us now employ the asymptotic expansion of the modified Bessel functions. For large values of ν, we have [42]

$$I_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{\mathrm{e}^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \Sigma_1 \qquad \Sigma_1 = \sum_{k=0}^\infty \frac{u_k}{\nu^k} \tag{3.5}$$

$$I'_\nu(\nu z) \sim \frac{1}{\sqrt{2\pi\nu}} \frac{\mathrm{e}^{\nu\eta}(1+z^2)^{\frac{1}{4}}}{z} \Sigma_2 \qquad \Sigma_2 = \sum_{k=0}^\infty \frac{v_k}{\nu^k} \tag{3.6}$$

$$K_\nu(\nu z) \sim \sqrt{\frac{\pi}{2\nu}} \frac{\mathrm{e}^{-\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \Sigma_3 \qquad \Sigma_3 = \sum_{k=0}^\infty (-1)^k \frac{u_k}{\nu^k} \tag{3.7}$$

$$K'_\nu(\nu z) \sim -\sqrt{\frac{\pi}{2\nu}} \frac{\mathrm{e}^{-\nu\eta}(1+z^2)^{\frac{1}{4}}}{z} \Sigma_4 \qquad \Sigma_4 = \sum_{k=0}^\infty (-1)^k \frac{v_k}{\nu^k} \tag{3.8}$$

where $\eta = \sqrt{1+z^2} + \ln[z/(1+\sqrt{1+z^2})]$. The first few coefficients u_k and v_k are listed in [42], while higher-order coefficients are immediate to obtain by using the recursion relations

$$u_{k+1}(t) = \tfrac{1}{2}t^2(1-t^2)u'_k(t) + \tfrac{1}{8}\int_0^t (1-5\tau^2)u_k(\tau)\,\mathrm{d}\tau \tag{3.9}$$

$$v_{k+1}(t) = u_{k+1}(t) - \tfrac{1}{2}t(1-t^2)u_k(t) - t^2(1-t^2)u'_k(t) \tag{3.10}$$

$$t = \frac{1}{\sqrt{1+z^2}} \qquad z = \frac{\sqrt{1-t^2}}{t}. \tag{3.11}$$

As we shall see explicitly in the following, the above behaviour of Bessel functions permits us to write

$$\ln P(\nu, z\nu) \sim \ln F(\nu, z) + \sum_{n=1}^N \frac{D_n(t)}{\nu^n} \tag{3.12}$$

an expression which is valid for large values of ν. The function F is related to the exponential factors in equations (3.5)–(3.8), while the coefficients $D_n(t)$ are related to Σ_k and are polynomials in t. More precisely

$$D_n(t) = \sum_{k=0}^{2n} x_{nk} t^{n+k}. \tag{3.13}$$

Note that when $b = \pm 1$, all x_{nk} with odd k vanish. Of course, F, D_n and x_{nk} depend on the specific problem under consideration. We will specialize them for every case.

The trick consists now in subtracting the asymptotic behaviour from the integrand function and in integrating the asymptotic part, with arbitrary s, exactly. We thus obtain

$$\zeta(s) = Z_0(s) + Z(s) + \sum_{n=-1}^{N} A_n(s). \tag{3.14}$$

Here,

$$Z_0(s) = \delta(D-2) d[0] \frac{\sin(\pi s)}{\pi} \int_0^\infty \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \ln P(0, z) \tag{3.15}$$

is the contribution due to $\nu_l = 0$, which is present only in two dimensions and has to be treated specifically for any case. $Z(s)$ represents all the other terms with the asymptotic contributions subtracted, that is

$$Z(s) = \frac{\sin(\pi s)}{\pi} \sum_{\nu_l > 0} d(l) \int_0^\infty \mathrm{d}z\, (z\nu)^{-2s} \frac{\partial}{\partial z} \left\{ \ln P(\nu_l, z\nu_l) - \ln F(\nu_l, z) - \sum_{n=1}^{N} \frac{D_n(t)}{\nu_l^n} \right\} \tag{3.16}$$

and A_n are the integrals of the asymptotic part. They read [27]

$$A_n(s) = -\frac{1}{\Gamma(s)} \zeta_{\mathcal{N}}(s + n/2) \sum_{k=0}^{2n} x_{nk} \frac{\Gamma\left(s + \frac{n+k}{2}\right)}{\Gamma\left(\frac{n+k}{2}\right)} \qquad n \geqslant 1 \tag{3.17}$$

$$\begin{aligned} A_{-1} + A_0 &= \frac{\sin(\pi s)}{\pi} \sum_{\nu_l > 0} d(l) \int_0^\infty \mathrm{d}z\, (z\nu_l)^{-2s} \frac{\partial}{\partial z} \ln[(z\nu_l)^{-b\nu_l} F(\nu_l, z)] \\ &= c_{-1}(s) \zeta_{\mathcal{N}}(s - 1/2) + c_0(s) \zeta_{\mathcal{N}}(s) \end{aligned} \tag{3.18}$$

$$\zeta_{\mathcal{N}}(s) = \sum_{\nu_l > 0} \mathrm{d}(l) \nu_l^{-2s}. \tag{3.19}$$

Equation (3.16) is convergent for $(D-2-N)/2 < \mathrm{Re}\ s < 1$, thus for our aim it is sufficient to subtract $N = D$ asymptotic terms. This means that with $N = D$ we can directly put $s = -1/2$ in equation (3.16) and perform the integral numerically.

As we shall see in the explicit examples, the base ζ-function, $\zeta_{\mathcal{N}}$, can be conveniently expressed in terms of the Barnes zeta function [44], defined as [45]

$$\begin{aligned} \zeta_{\mathcal{B}}(s, a; d) &= \sum_{\vec{m}=0}^{\infty} \frac{1}{(a + m_1 + \cdots + m_d)^s} = \sum_{n=0}^{\infty} e_n(d) (a+n)^{-s} \\ e_n(d) &= \frac{(d+n-1)!}{n!(d-1)!} \end{aligned}$$

for $\mathrm{Re}\ s > d$. Obviously, there is an expansion of the kind

$$e_n(d) = \sum_{\alpha=0}^{d-1} g_\alpha(d) (a+n)^\alpha$$

and this yields the expansion of the Barnes zeta function in terms of Hurwitz zeta functions [45, 46]

$$\zeta_{\mathcal{B}}(s, a; d) = \sum_{\alpha=0}^{d-1} g_\alpha(d)\zeta_H(s-\alpha, a). \tag{3.20}$$

For example, for $d = 2$, we trivially obtain

$$\zeta_{\mathcal{B}}(s, a; 2) = \zeta_H(s-1, a) + (1-a)\zeta_H(s, a).$$

One can show that the $g_\alpha(d)$ are connected with the generalized Bernoulli polynomials [47]. This allows us to determine, in a direct way, the residues and finite parts of the Barnes zeta function of the problem at hand. As a result, the asymptotic contributions in (3.14) are readily computed.

4. The scalar field

The field equation for this case reads

$$-\Delta\phi_k(\vec{x}) = \lambda_k\phi_k(\vec{x}) \tag{4.1}$$

and has to be supplemented with Dirichlet or Robin boundary conditions. Here, Δ is the Laplace operator inside or outside the $D = (d+1)$-dimensional ball and we impose Dirichlet $(\phi(\vec{x})|_{|\vec{x}|=R} = 0)$ or Robin $([\alpha\phi(\vec{x})| + \phi'(\vec{x})]|_{|\vec{x}|=R} = 0)$ boundary conditions.

In polar coordinates the solutions are

$$\phi_{l,m,n}(r, \Omega) = r^{1-D/2} f_{\nu_1}(\omega_{l,n} r) Y_{l+D/2}(\Omega)$$

with $\nu_l = l + (D-2)/2$, the $f_\nu(r)$ being Bessel functions and the $Y_{l+D/2}(\Omega)$ hyperspherical harmonics [43].

4.1. Scalar field with Dirichlet boundary conditions inside a spherical shell

In this case, the f_ν are Bessel functions of the first kind and thus the eigenvalues $\lambda_{l,n} = \omega_{l,n}^2$ are defined through

$$J_{\nu_l}(\omega_{l,n} R) = 0$$

and have degeneracies given by $d(l) = (2l + d - 1)\frac{(l+d-2)!}{l!(d-1)!}$. From the last equation, it easily follows that [27, 44]

$$\zeta_{\mathcal{N}} = \zeta_{\mathcal{B}}\left(2s, \frac{d+1}{2}; d\right) + \zeta_{\mathcal{B}}\left(2s, \frac{d-1}{2}; d\right). \tag{4.2}$$

In this case,

$$\tilde{P}(\tilde{Z}_\nu(k), \tilde{Z}'_\nu(k)) = J_\nu(k) \qquad P(\nu, z) = I_\nu(z) \tag{4.3}$$

and, as a consequence,

$$F(\nu, z) = \frac{1}{\sqrt{2\pi\nu}} \frac{\mathrm{e}^{\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \tag{4.4}$$

$$\ln \Sigma_1 \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n}. \tag{4.5}$$

The asymptotic contributions have been calculated to be [27]

$$\begin{aligned} A_{-1}(s) &= \frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)}\, \zeta_{\mathcal{N}}(s-1/2) \\ A_0(s) &= -\tfrac{1}{4}\, \zeta_{\mathcal{N}}(s). \end{aligned} \tag{4.6}$$

The ζ-function for the present situation is obtained by means of equations (3.13)–(3.17) with the definitions above.

As already anticipated in the previous section, in two dimensions we have an aditional contribution that has to be computed explicitly. With this aim, we recall that, for large z,

$$I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\left\{1 + \frac{1}{8z} + \mathcal{O}\big(z^{-2}\big)\right\}$$

and thus we can write

$$Z_0(s) = \frac{\sin(\pi s)}{\pi}\left\{\int_0^1 dz\, z^{-2s}\frac{\partial}{\partial z}\ln I_0(z) + \int_1^\infty dz\, z^{-2s}\left[\frac{\partial}{\partial z}\ln I_0(z) - 1 + \frac{1}{2z} + \frac{1}{8z^2}\right]\right.$$
$$\left. - \frac{1}{4s} + \frac{1}{2(s-1/2)} - \frac{1}{16(s+1/2)}\right\} \tag{4.7}$$

where the poles at $s = \pm 1/2$ are shown explicitly. The integrals are now convergent for $s = -1/2$ and can be computed numerically.

4.2. *Scalar field with Dirichlet boundary conditions outside a spherical shell*

Now the radial parts of the solutions are Bessel functions of the third kind (Hankel functions), while ν_l and $d(l)$ remain the same. Thus, we have

$$\nu_l = l + \frac{D-2}{2} \tag{4.8}$$

$$d(l) = (2l + d - 1)\frac{(l+d-2)!}{l!(d-1)!} \tag{4.9}$$

$$\zeta_{\mathcal{N}} = \zeta_{\mathcal{B}}\left(2s, \frac{d+1}{2}; d\right) + \zeta_{\mathcal{B}}\left(2s, \frac{d-1}{2}; d\right) \tag{4.10}$$

$$P(\nu, z) = K_\nu(z) \tag{4.11}$$

$$F(\nu, z) = \sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta}}{(1+z^2)^{\frac{1}{4}}} \tag{4.12}$$

$$\ln \Sigma_3 \sim \sum_{n=1}^{\infty}\frac{D_n(t)}{\nu^n} \tag{4.13}$$

$$A_{-1}(s) = -\frac{1}{4\sqrt{\pi}}\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)}\,\zeta_{\mathcal{N}}(s - 1/2) \tag{4.14}$$

$$A_0(s) = -\tfrac{1}{4}\,\zeta_{\mathcal{N}}(s). \tag{4.15}$$

Owing to the particular relation between Σ_1 and Σ_3, the coefficients $D_n(t)$ differ from the corresponding coefficients one has in the internal case just in the trivial factor $(-1)^n$. The same holds also for the quantities $A_n(s)$.

In two dimensions we have to also consider the contribution due to $\nu = 0$, which can be obtained with the same arguments as in the previous case, equation (4.7). The result is

$$Z_0(s) = \frac{\sin(\pi s)}{\pi}\left\{\int_0^1 dz\, z^{-2s}\frac{\partial}{\partial z}\ln K_0(z) + \int_1^\infty dz\, z^{-2s}\left[\frac{\partial}{\partial z}\ln K_0(z) + 1 + \frac{1}{2z} - \frac{1}{8z^2}\right]\right.$$
$$\left. - \frac{1}{4s} - \frac{1}{2(s-1/2)} + \frac{1}{16(s+1/2)}\right\}. \tag{4.16}$$

The numerical results corresponding to the ζ-functions inside and outside the shell and the total Casimir energy are reported in table 1 for the choices $D = 2, \ldots, 9$. For the interior space for $D = 2$ and 3 as well as for $D = 3$ and the exterior space, our results agree with [22]. For the whole space in $D = 3$ the result is given in [23, 39].

Table 1. *Scalar field with Dirichlet boundary conditions.* Values of the zeta function at $s = -1/2$, for the inside and the outside regions of a spherical shell, and values of the Casimir energy. The presence of the cutoff ϵ for all even dimensions is to be noted. In such cases, the Casimir energy is divergent and needs to be renormalized.

D	$\zeta(-1/2)$ inside	$\zeta(-1/2)$ outside	Casimir energy
2	$+0.009\,8540 - 0.003\,9062/\epsilon$	$-0.008\,4955 - 0.003\,9062/\epsilon$	$+0.000\,6793 - 0.003\,9062/\epsilon$
3	$+0.008\,8920 + 0.001\,0105/\epsilon$	$-0.003\,2585 - 0.001\,0105/\epsilon$	$+0.002\,8168$
4	$-0.001\,7939 + 0.000\,2670/\epsilon$	$+0.000\,4544 + 0.000\,2670/\epsilon$	$-0.000\,6698 + 0.000\,2670/\epsilon$
5	$-0.000\,9450 - 0.000\,1343/\epsilon$	$+0.000\,3739 + 0.000\,1343/\epsilon$	$-0.000\,2856$
6	$+0.000\,2699 - 0.000\,0335/\epsilon$	$-0.000\,0611 - 0.000\,0335/\epsilon$	$+0.000\,1044 - 0.000\,0335/\epsilon$
7	$+0.000\,1371 + 0.000\,0214/\epsilon$	$-0.000\,0555 - 0.000\,0214/\epsilon$	$+0.000\,040\,8$
8	$-0.000\,0457 + 5.228 \times 10^{-6}/\epsilon$	$+0.000\,0101 + 5.228 \times 10^{-6}/\epsilon$	$-0.000\,0178 + 5.228 \times 10^{-6}/\epsilon$
9	$-0.000\,0230 - 3.769 \times 10^{-6}/\epsilon$	$+0.000\,0094 + 3.769 \times 10^{-6}/\epsilon$	$-0.000\,0068$

4.3. *Scalar field with Robin boundary conditions inside a spherical shell*

In the case of Robin boundary conditions the radial part of the solution is a combination of Bessel functions with derivatives. For the interior case we have Bessel functions of the first kind and their eigenvalues are determined through

$$\left(1 - \frac{D}{2} - \beta\right) J_{\nu_l}(\omega_{l,n}) + \omega_{l,n} J'_{\nu_l}(\omega_{l,n}) = 0. \tag{4.17}$$

Here we have put $\alpha = 1 - D/2 - \beta$ and, in the spirit of section 3, we have to restrict ourselves to the case $\beta \geqslant 1 - D/2 - \nu_0$. The choice $\beta = 0$ represents Neumann boundary conditions.

Also for this case ν_l, $d(l)$ and $\zeta_{\mathcal{N}}$ are given by equations (4.8)–(4.10), now with

$$P(\nu, z) = \left(1 - \frac{D}{2} - \beta\right) I_\nu(z) + z I'_\nu(z)$$

$$F(\nu, z) = \sqrt{\frac{\nu}{2\pi}} \mathrm{e}^{\nu\eta}(1 + z^2)^{\frac{1}{4}}$$

$$\ln\left(\frac{1 - D/2 - \beta}{\nu} t\Sigma_1 + \Sigma_2\right) \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n}$$

$$A_{-1}(s) = \frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s+1)} \zeta_{\mathcal{N}}(s - 1/2)$$

$$A_0(s) = \tfrac{1}{4} \zeta_{\mathcal{N}}(s).$$

In two dimensions we have to consider also the contribution

$$Z_0(s) = \frac{\sin(\pi s)}{\pi} \left\{ \int_0^1 \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \ln(\alpha I_0(z) + z I'_0(z)) \right. + \int_1^\infty \mathrm{d}z\, z^{-2s} \left[\frac{\partial}{\partial z} \ln(\alpha I_0(z) + z I'_0(z)) - 1 - \frac{1}{2z} - \left(\frac{3}{8} - \alpha\right) \frac{1}{z^2} \right] \left. + \frac{1}{4s} + \frac{1}{2(s - 1/2)} + \left(\frac{3}{8} - \alpha\right) \frac{1}{2(s + 1/2)} \right\}.$$

4.4. *Scalar field with Robin boundary conditions outside the spherical shell*

As for Dirichlet, the only difference between the interior and the exterior case consists in the replacement of Bessel functions with Hankel functions. Equations (4.8)–(4.10) are valid again,

Table 2. *Scalar field with Neumann boundary conditions (or Robin with the choice $\beta = 0$).* Values of the zeta function at $s = -1/2$, for the inside and the outside regions of a spherical shell, and corresponding values of the Casimir energy.

D	$\zeta(-1/2)$ inside	$\zeta(-1/2)$ outside	Casimir energy
2	$-0.344\,6767 - 0.019\,5312/\epsilon$	$-0.021\,5672 - 0.019\,5312/\epsilon$	$-0.183\,1220 - 0.019\,5312/\epsilon$
3	$-0.459\,7174 - 0.035\,3678/\epsilon$	$+0.012\,0743 + 0.035\,3678/\epsilon$	$-0.223\,8215$
4	$-0.515\,3790 - 0.044\,7159/\epsilon$	$-0.006\,0394 - 0.044\,7159/\epsilon$	$-0.260\,7092 - 0.044\,7159/\epsilon$
5	$-0.555\,2071 - 0.048\,9213/\epsilon$	$+0.003\,0479 + 0.048\,9213/\epsilon$	$-0.276\,0796$
6	$-0.594\,9395 - 0.051\,3727/\epsilon$	$-0.012\,8321 - 0.051\,3727/\epsilon$	$-0.303\,8858 - 0.051\,3727/\epsilon$

while

$$
\begin{aligned}
P(\nu, z) &= \left(1 - \frac{D}{2} - \beta\right) K_\nu(z) + z K'_\nu(z) \\
F(\nu, z) &= \sqrt{\frac{\pi\nu}{2}} \mathrm{e}^{-\nu\eta} (1 + z^2)^{\frac{1}{4}} \\
\ln\left(\frac{1 - D/2 - \beta}{\nu}\, t\Sigma_3 - \Sigma_4\right) &\sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n} \\
A_{-1}(s) &= -\frac{1}{4\sqrt{\pi}} \frac{\Gamma\left(s - \frac{1}{2}\right)}{\Gamma(s + 1)}\, \zeta_{\mathcal{N}}\,(s - 1/2) \\
A_0(s) &= \tfrac{1}{4}\, \zeta_{\mathcal{N}}(s).
\end{aligned}
$$

For the $\nu = 0$ contribution, we have in this case

$$
\begin{aligned}
Z_0(s) = \frac{\sin(\pi s)}{\pi} \Bigg\{ & \int_0^1 \mathrm{d}z\; z^{-2s} \frac{\partial}{\partial z} \ln(\alpha K_0(z) + z K'_0(z)) \\
& + \int_1^\infty \mathrm{d}z\; z^{-2s} \left[\frac{\partial}{\partial z} \ln(\alpha K_0(z) + z K'_0(z)) + 1 - \frac{1}{2z} + \left(\frac{3}{8} - \alpha\right) \frac{1}{z^2} \right] \\
& + \frac{1}{4s} - \frac{1}{2(s - 1/2)} - \left(\frac{3}{8} - \alpha\right) \frac{1}{2(s + 1/2)} \Bigg\}.
\end{aligned}
$$

All numerical results corresponding to Neumann boundary conditions (or Robin ones with $\beta = 0$) are exhibited in table 2. For $D = 2$ the result is given in [22], for $D = 3$ in [23].

5. Spinor field on the D-dimensional ball: bag boundary conditions

We now consider spinor fields, see [37, 48]. The eigenvalue Dirac equation on the Euclidean D-ball is

$$
-\mathrm{i}\Gamma^\mu \nabla_\mu \psi_\pm = \pm k \psi_\pm \qquad \Gamma^{(\mu}\Gamma^{\nu)} = g^{\mu\nu} \tag{5.1}
$$

and the nonzero modes are separated in polar coordinates, $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega^2$, in standard fashion to be regular at the origin (C and A are radial normalization factors),

$$
\begin{aligned}
\psi_\pm^{(+)} &= \frac{A}{r^{(D-2)/2}} \begin{pmatrix} \mathrm{i} J_{n+D/2}(kr)\, Z_+^{(n)}(\Omega) \\ \pm J_{n+D/2-1}(kr)\, Z_+^{(n)}(\Omega) \end{pmatrix} \\
\psi_\pm^{(-)} &= \frac{C}{r^{(D-2)/2}} \begin{pmatrix} \pm J_{n+D/2-1}(kr)\, Z_-^{(n)}(\Omega) \\ i\, J_{n+D/2}(kr)\, Z_-^{(n)}(\Omega) \end{pmatrix}.
\end{aligned} \tag{5.2}
$$

Here the $Z_{\pm}^{(n)}(\Omega)$ are well known spinor modes on the unit $(D-1)$-sphere (some modern references are [49–51]) satisfying the intrinsic equation

$$-\mathrm{i}\gamma^j \tilde{\nabla}_j Z_{\pm}^{(n)} = \pm\lambda_n Z_{\pm}^{(n)} \tag{5.3}$$

where

$$\lambda_n = n + \frac{D-1}{2} \qquad n = 0, 1, \ldots.$$

For $D \geqslant 2$, each eigenvalue is greater than or equal to one-half and has degeneracy

$$\tfrac{1}{2} d_{\mathrm{s}} \begin{pmatrix} D+n-2 \\ n \end{pmatrix}.$$

The dimension, d_{s}, of ψ-spinor space is $2^{D/2}$ if D is even. For odd D it is $2^{(D+1)/2}$ and has been doubled in order to implement the boundary conditions. The projected γ-matrices are given by

$$\Gamma^r = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \qquad \Gamma^j = \begin{pmatrix} \mathbf{0} & \mathrm{i}\gamma^j \\ -\mathrm{i}\gamma^j & \mathbf{0} \end{pmatrix} \qquad \Gamma^5 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}. \tag{5.4}$$

5.1. Spinor field inside a spherical shell: bag boundary conditions

For bag—also called mixed—boundary conditions, we apply $P_+\psi = 0$ at $r = 1$, where the projection is given by

$$P_+ = \tfrac{1}{2}\left(\mathbf{1} - \mathrm{i}\Gamma^5\Gamma^\mu n_\mu\right) \tag{5.5}$$

in terms of the inward normal n_μ. For the geometry of the ball

$$P_+ = \frac{1}{2}\begin{pmatrix} \mathbf{1} & \mathrm{i}\mathbf{1} \\ -\mathrm{i}\mathbf{1} & \mathbf{1} \end{pmatrix}$$

and so for $\psi_{\pm}^{(+)}$,

$$J_{n+D/2}(k) = \mp J_{n+D/2-1}(k)$$

and for $\psi_{\pm}^{(-)}$,

$$J_{n+D/2-1}(k) = \mp J_{n+D/2}(k) \qquad n = 0, 1, 2, \ldots.$$

Thus, taking $\nu_n = n + (D-2)/2$, the implicit eigenvalue equation is as in [52]

$$J_\nu^2(k) - J_{\nu+1}^2(k) = 0 \tag{5.6}$$

while the degeneracies are

$$d(n) = d_{\mathrm{s}} \begin{pmatrix} D+n-2 \\ D-2 \end{pmatrix}. \tag{5.7}$$

In two dimensions the degeneracy is just two.

In summary, all the relevant functions for this case are

$$
\begin{aligned}
&\nu_n = n + \frac{D-2}{2} \\
&d(n) = d_s \frac{(n+D-2)!}{n!(D-2)!} \qquad d(n) = 2 \qquad \text{for} \quad D=2 \\
&\zeta_{\mathcal{N}}(s) = d_s \zeta_{\mathcal{B}}(2s, D/2-1; d) \qquad \zeta_{\mathcal{N}}(s) = 2\zeta_{\mathrm{R}}(2s) \qquad \text{for} \quad D=2 \\
&P(\nu, z) = I_\nu^2(z) + I_{\nu+1}^2(z) \\
&F(\nu, z) = \frac{(1-t)\mathrm{e}^{2\nu\eta}(1+z^2)^{\frac{1}{2}}}{\pi\nu z^2} \\
&\ln\left(\frac{1}{2(1-t)}\left[\Sigma_1^2 + \Sigma_2^2 - 2t\Sigma_1\Sigma_2\right]\right) \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n} \\
&A_{-1}(s) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)} \zeta_{\mathcal{N}}(s-1/2) \\
&A_0(s) = -\frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s+1)} \zeta_{\mathcal{N}}(s)
\end{aligned}
\tag{5.8}
$$

with ζ_{R} the Riemann ζ-function.

The contribution of $\nu = 0$, which we have in two dimensions, reads here

$$
\begin{aligned}
Z_0(s) = \frac{\sin(\pi s)d[0]}{\pi} \Bigg\{ &\int_0^1 \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \ln(I_0^2(z) + I_1^2(z)) \\
&+ \int_1^\infty \mathrm{d}z\, z^{-2s} \left[\frac{\partial}{\partial z} \ln(I_0^2(z) + I_1^2(z)) - 2 + \frac{1}{z} - \frac{1}{4z^2}\right] \\
&- \frac{1}{2s} + \frac{1}{(s-1/2)} + \frac{1}{8(s+1/2)} \Bigg\}.
\end{aligned}
$$

5.2. *Spinor field outside a spherical shell: bag boundary conditions*

As in the scalar cases, we must simply replace Bessel with Hankel functions. Equations (5.8) and (5.9) provide some quantities needed in the computation, while for the rest we obtain

$$
\begin{aligned}
&P(\nu, z) = K_\nu^2(z) + K_{\nu+1}^2(z) \\
&F(\nu, z) = \frac{4\nu(1+t)\mathrm{e}^{-2\nu\eta}(1+z^2)^{\frac{1}{2}}}{\pi z^2} \\
&\ln\left(\frac{1}{2(1+t)}\left[\Sigma_3^2 + \Sigma_4^2 + 2t\Sigma_3\Sigma_4\right]\right) \sim \sum_{n=1}^{\infty} \frac{D_n(t)}{\nu^n} \\
&A_{-1}(s) = -\frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s+1)} \zeta_{\mathcal{N}}(s-1/2) \\
&A_0(s) = \frac{1}{2\sqrt{\pi}} \frac{\Gamma\left(s+\frac{1}{2}\right)}{\Gamma(s+1)} \zeta_{\mathcal{N}}(s).
\end{aligned}
$$

In the same way, for $\nu = 0$ we obtain

$$
\begin{aligned}
Z_0(s) = \frac{\sin(\pi s)d[0]}{\pi} \Bigg\{ &\int_0^1 \mathrm{d}z\, z^{-2s} \frac{\partial}{\partial z} \ln\left(K_0^2(z) + K_1^2(z)\right) \\
&+ \int_1^\infty \mathrm{d}z\, z^{-2s} \left[\frac{\partial}{\partial z} \ln\left(K_0^2(z) + K_1^2(z)\right) + 2 + \frac{1}{z} + \frac{1}{4z^2}\right] \\
&- \frac{1}{2s} - \frac{1}{(s-1/2)} - \frac{1}{8(s+1/2)} \Bigg\}.
\end{aligned}
$$

Table 3. *Massless spinor field with mixed boundary conditions*. Values of the zeta function at $s = -1/2$, for the inside and the outside regions of a spherical shell, and values of the Casimir energy.

D	$\zeta(-1/2)$ inside	$\zeta(-1/2)$ outside	Casimir energy
2	$-0.005\,8312 + 0.007\,8125/\epsilon$	$+0.021\,3677 + 0.007\,8125/\epsilon$	$-0.007\,7683 - 0.007\,8125/\epsilon$
3	$-0.060\,5944 - 0.005\,0525/\epsilon$	$+0.019\,8217 + 0.005\,0525/\epsilon$	$+0.020\,3863$
4	$+0.005\,9074 - 0.002\,8381/\epsilon$	$-0.010\,1965 - 0.002\,8381/\epsilon$	$+0.002\,1445 + 0.002\,8381/\epsilon$
5	$+0.025\,0447 + 0.002\,5110/\epsilon$	$-0.008\,9912 - 0.002\,5110/\epsilon$	$-0.008\,0268$
6	$-0.003\,0244 + 0.001\,1715/\epsilon$	$+0.004\,6183 + 0.001\,1715/\epsilon$	$-0.000\,7969 - 0.001\,1715/\epsilon$
7	$-0.010\,8618 - 0.001\,1745/\epsilon$	$+0.004\,0247 + 0.001\,1745/\epsilon$	$+0.003\,4186$

The numerical results for spin 1/2 with bag boundary conditions are given in table 3. The $D = 3$ result is that found already by Milton [15] (albeit with far less precision).

5.3. Spinor field with global spectral boundary conditions

We shall now obtain the results for spectral boundary conditions [37, 48]. Such boundary conditions are imposed by setting equal to zero, at $r = 1$, the negative (positive) Z-modes of the positive (negative) chirality parts of ψ, the rest of the modes remaining free.

Roughly speaking, spectral conditions amount to requiring that zero modes of (5.1) should be square integrable on the elongated manifold obtained from the ball by extending the narrow collar (of the approximate product metric $\mathrm{d}r^2 + \mathrm{d}\Omega^2$) just inside the surface, to values of r ranging from 1 to ∞. This will be so if the modes of $A = \Gamma^r\Gamma^a\nabla_a$ with *negative* eigenvalues are suppressed at the boundary (e.g. [53–61]).

From (5.4) and (5.3), the boundary operator is $A_0 = \Gamma^r\Gamma^a\nabla_a\big|_{r=1}$ and has for eigenstates

$$A_0\begin{pmatrix} Z_+^{(n)} \\ Z_-^{(n)} \end{pmatrix} = \lambda_n \begin{pmatrix} Z_+^{(n)} \\ Z_-^{(n)} \end{pmatrix} \qquad A_0\begin{pmatrix} Z_-^{(n)} \\ Z_+^{(n)} \end{pmatrix} = -\lambda_n \begin{pmatrix} Z_-^{(n)} \\ Z_+^{(n)} \end{pmatrix}. \tag{5.9}$$

Thus, from (5.2) we see that the negative modes of A_0 are associated with the radial factor $J_{n+D/2-1}(kr)$. Taking ν as before, $\nu = n + (D-2)/2$, the implicit eigenvalue equation reads

$$J_\nu(k) = 0.$$

The degeneracy for each eigenvalue is

$$d(n) = 2d_\mathrm{s}\binom{n+D-2}{D-2} \qquad d(n) = 4 \qquad \text{for} \quad D = 2. \tag{5.10}$$

The relevant boundary zeta function now reads

$$\zeta_\mathcal{N}(s) = 2d_\mathrm{s}\zeta_\mathcal{B}(2s, D/2-1; d) \qquad \zeta_\mathcal{N}(s) = 4\zeta_\mathrm{R}(2s) \qquad \text{for} \quad D = 2. \tag{5.11}$$

As we see, apart from the degeneracy of the eigenvalues and the relation between $\zeta_\mathcal{N}$ and the Barnes ζ-function, the rest of the argumentation is identical to that for the scalar case with Dirichlet boundary conditions. Thus, equations (4.3)–(4.7) remain valid once the above definitions are used.

For the exterior space we have to employ equations (5.10) and (5.11) in equations (4.11)–(4.16), but it has to be noted that here, in contrast with the interior case, $\nu_l = l + D/2$, as a result of the normal vector changing its sign. This means that there is no $\nu_l = 0$ contribution.

The numerical results for this case are listed in table 4, for $D = 2, \ldots, 9$.

Table 4. *Massless spinor field with global spectral boundary conditions*. Values of the zeta function at $s = -1/2$, for the inside and the outside regions of a spherical shell, and corresponding values of the Casimir energy.

D	$\zeta(-1/2)$ inside	$\zeta(-1/2)$ outside	Casimir energy
2	$-0.009\,3152 + 0.031\,9762/\epsilon$	$+0.010\,0172 + 0.031\,9762/\epsilon$	$-0.000\,3510 - 0.031\,9762/\epsilon$
3	$-0.171\,0212 - 0.003\,7705/\epsilon$	$+0.001\,9763 + 0.003\,7705/\epsilon$	$+0.084\,5225$
4	$+0.008\,2635 - 0.011\,8316/\epsilon$	$-0.004\,0473 - 0.011\,8316/\epsilon$	$-0.002\,1081 + 0.0118\,316/\epsilon$
5	$+0.068\,0217 + 0.001\,9471/\epsilon$	$-0.000\,9007 - 0.001\,9471/\epsilon$	$+0.033\,5605$
6	$-0.004\,2224 + 0.004\,9069/\epsilon$	$+0.001\,7603 + 0.004\,9069/\epsilon$	$+0.001\,2311 - 0.004\,9069/\epsilon$
7	$-0.029\,0717 - 0.000\,9256/\epsilon$	$+0.000\,3983 + 0.000\,9256/\epsilon$	$+0.014\,3367$
8	$+0.002\,0298 - 0.002\,1417/\epsilon$	$-0.000\,7907 - 0.002\,1417/\epsilon$	$-0.000\,6196 + 0.002\,1417/\epsilon$
9	$+0.012\,8994 + 0.000\,4353/\epsilon$	$-0.000\,1787 - 0.000\,4353/\epsilon$	$-0.006\,3604$

Table 5. *Electromagnetic field in a perfectly conducting spherical shell*. Values of the zeta function at $s = -1/2$, for the inside and the outside regions of a spherical shell, and corresponding values of the Casimir energy. Note that in even dimensions, in contrast with the scalar field, the divergences arising from the inside and the outside energies are different. This is due to the fact that (only in even dimensions) the $l = 0$ mode explicitly contributes to the poles of the ζ-function, such contribution being absent from the scalar case.

D	$\zeta(-1/2)$ inside	$\zeta(-1/2)$ outside	Casimir energy
2	$-0.344\,6767 - 0.019\,5312/\epsilon$	$-0.021\,5672 - 0.019\,5312/\epsilon$	$-0.183\,1220 - 0.019\,5312/\epsilon$
3	$+0.167\,8471 + 0.008\,0841/\epsilon$	$-0.075\,4938 - 0.008\,0841/\epsilon$	$+0.046\,1767$
4	$+0.500\,8593 + 0.023\,1719/\epsilon$	$-0.194\,2082 - 0.056\,4056/\epsilon$	$+0.153\,3255 - 0.033\,2337/\epsilon$
5	$+1.046\,3255 + 0.183\,8665/\epsilon$	$-0.298\,1425 - 0.183\,8665/\epsilon$	$+0.374\,0915$

6. Electromagnetic field in a perfectly conducting spherical shell

The Casimir energy of the electromagnetic field is, essentially, the sum of a Dirichlet and of a Robin scalar field (with a specific value for β, see equation (4.17)), the only difference being that the angular momentum $l = 0$ is to be omitted. An exception is $D = 2$, where the vector Casimir effect consists of only the transverse magnetic mode contributions. Being precise, in the interior of the shell one has for the transverse electric (TE)—respectively for the transverse magnetic (TM) modes—the following boundary conditions [1, 62]:

$$\begin{aligned} &r^{1-D/2} J_{\nu_l}(\omega_{l,n} r)|_{r=R} = 0 \qquad \text{for} \quad \text{TE modes} \\ &\left[(D/2-1) J_{\nu_l}(\omega_{l,n} r) + \omega_{l,n} J'_{\nu_l}(\omega_{l,n} r)\right]|_{r=R} = 0 \qquad \text{for} \quad \text{TM modes.} \end{aligned} \tag{6.1}$$

The condition for the TM modes is of Robin type with $\beta = 2 - D$. Since the $l = 0$ mode has to be omitted, the minimum eigenvalue in this case is $\mu_1 = D/2$ and therefore we can apply the method for any $\beta = 2 - D > 1 - D$. Thus, in order to obtain the Casimir energy of the electromagnetic field, we must simply repeat the computation of section 4 for Dirichlet and Robin boundary conditions with $\beta = 2 - D$ and add them up. We have to exclude everywhere the $l = 0$ mode and this means that also the base ζ-function is slightly modified, in the way

$$\zeta_{\mathcal{N}}(s) = \zeta_{\mathcal{B}}\left(2s, \frac{d+1}{2}; d\right) + \zeta_{\mathcal{B}}\left(2s, \frac{d-1}{2}; d\right) - \left(\frac{d-1}{2}\right)^{-2s}. \tag{6.2}$$

The results for the electromagnetic field are summarized in table 5. $D = 2$ is the Neumann result, $D = 3$ is the well known figure first obtained by Boyer [1] and later recalculated in [63, 64].

7. Discussion and conclusions

In [26], two of the present authors developed a new, seminal approach for finding representations of the zeta function associated with the Laplace operator on the D-dimensional ball. At that stage, dimension by dimension was considered, but soon a refined and generalized technique was provided in subsequent works [27, 37]. Making use of the Barnes zeta function [45], dimension can easily be dealt with as a parameter and several different fields can also be treated on the same footing. The representations derived are valid for all values of the complex parameter s and it depends on the practitioner's needs or wishes at which values of s the zeta function is to be evaluated. In previous work our concern was of a more mathematical nature and we considered function values and residues appropriate to find heat-kernel coefficients [26, 27, 37], as well as the derivative at $s = 0$ [30]. Since then, a number of proclaimed 'new' methods have been developed in the literature.

Our aim in this paper has been to show explicitly that Casimir energies for the large family of the more usual configurations can be obtained in fact from *general* formulae, also in quite non-trivial situations, where the boundaries are not flat plates, the fields are spinorial (rather than scalar) and also when the boundary conditions are very general and rather involved. We have gone far beyond previous work in that here we are no longer restricted to a very specific field in a specific dimension with a specific boundary condition, but give general formulae for basically any possible situation that can arise in practice, involving spherically symmetric boundaries.

Some comments on the precision and accuracy of the numerical procedure employed are in order. It is clear from the analysis developed in the previous sections that a numerical evaluation of the asymptotic terms $A_i(s)$ to *any* desired accuracy is immediate, using the formulae given there. These contributions are always represented by known special functions and using available programs, such as Mathematica, the accuracy with which these are calculated is readily obtained. Imposing accuracies of, for example, 10^{-20} or more, we obtain results in negligible cpu time.

The only problem (if any) with the numerical analysis is the computation of $Z(s)$, equation (3.14). It is twofold. On one hand, the integration, up to *infinity*, of the combination of Bessel functions is not strictly possible, using the exact form of the Bessel functions. On the other hand, the angular summation, up to *infinity*, cannot be performed exactly. For large angular momenta, the Bessel functions take a rather complicated form, which renders exact summation not possible. For that reason, the following procedure has been applied throughout (the error bounds given below are for Dirichlet boundary conditions, but very similar relations hold for the other conditions considered).

We have dealt with the infinite integration as follows. The main contributions always originate from small values of z, and thus we split the integral into

$$\int_0^B \mathrm{d}z + \int_B^\infty \mathrm{d}z.$$

Whereas in the first integral the Bessel functions themselves are used for the integration, in the second integral their asymptotic expansion for large arguments is employed. The value of B is computed with the help of an adaptative procedure, such that the integrand and its asymptotic expansion differ, at B, by less than, say 10^{-12}. Typically $B = 10$ is already sufficient. Given that the asymptotic of the Bessel functions is a simple polynomial in powers of $(1/z)$, the integration up to infinity is very easily done.

Let us now assume that the contribution of the first L angular momenta has been calculated as described. In order to obtain a numerical approximation for the angular momentum sum, from $L+1$ to infinity, we proceed as follows. The idea is that, for sufficiently large values of L, the integrand can be replaced by its uniform asymptotic expression. For Dirichlet boundary conditions this amounts to going through the following steps:

$$\begin{aligned}
Z_{L+1}^{\text{int}} &\equiv -\frac{1}{\pi}\sum_{l=L+1}^{\infty} d(l)\nu \int_0^\infty \mathrm{d}z \left[\ln I_\nu(\nu z) - \ln \frac{\mathrm{e}^{\nu\eta}}{\sqrt{2\pi\nu}(1+z^2)^{1/4}} - \sum_{n=1}^{N}\frac{D_n(t)}{\nu^n}\right] \\
&\sim -\frac{1}{\pi}\Bigg[\left(\int_0^\infty \mathrm{d}z\ D_{N+1}(t)\right)\sum_{l=L+1}^{\infty} d(l)\nu^{-N} \\
&\quad +\left(\int_0^\infty \mathrm{d}z\ D_{N+2}(t)\right)\sum_{l=L+1}^{\infty} d(l)\nu^{-N-1} + \cdots\Bigg] \\
&= -\frac{1}{\pi}\Bigg[\left(\int_0^\infty \mathrm{d}z\ D_{N+1}(t)\right)\left(\zeta_{\mathcal{N}}(N/2) - \sum_{l=0}^{L} d(l)\nu^{-N}\right) \\
&\quad +\left(\int_0^\infty \mathrm{d}z\ D_{N+2}(t)\right)\left(\zeta_{\mathcal{N}}((N+1)/2) - \sum_{l=0}^{L} d(l)\nu^{-N-1}\right) + \cdots\Bigg].
\end{aligned} \tag{7.1}$$

Again, the integrals over the uniform asymptotics are simple and can be performed analytically. In this way, a closed expression for the approximation is found, the value of L being again determined by an adaptative procedure. By definition, the difference $Z_L^{\text{int}} - Z_{L+1}^{\text{int}}$ is equal to the contribution originating from $l = L$. The value of L is determined such that the difference $Z_L^{\text{int}} - Z_{L+1}^{\text{int}}$, obtained from (7.1), agrees up to say 10^{-10} with the contribution from $l = L$ calculated previously. Depending on the dimension, the values of L range from 6 (for $D = 9$) to 49 (for $D = 3$).

In summary, as explained, this procedure takes fully into account the integrals of infinite range as well as the summation up to infinity. The error bounds can thus be imposed at will in the single steps and this guarantees that the results given are always numerically precise, up to *any* pre-established digit. To our knowledge, this does not apply to any other method.

In the cases when partial results were known, we have compared our numbers with these while improving always, by several digits, such known values and deriving, for the first time, many new ones, for different fields (e.g. results for the exterior space in the case of the electromagnetic field) and different boundary conditions (e.g. for spectral boundary conditions, and for bag boundary conditions in any dimension). For the scalar field with Dirichlet boundary conditions we have re-obtained, in particular, the known result that for even D the energy is divergent [39]. Here it still remains unclear whether there may be a natural way to obtain, unambiguously, a finite answer with physical sense for this case. In odd dimension, $D = 2n - 1$, the sign of the Casimir energy seems to be determined by the sign of $(-1)^n$. For even dimension, $D = 2n$, one also finds the alternating structure $(-1)^{n+1}$ for the *finite* part of the Casimir energy; however, its interpretation is unclear due to the presence of the pole. Similar comments hold for the interior and exterior contributions separately, with the same problems of interpretation. For Neumann boundary conditions, in all the dimensions calculated, the Casimir energy is negative. Similarly, one can describe the results summarized in tables 3–5. In all cases we have been able to obtain *general*, *highly accurate* expressions, which, by fixing some parameters, provide us with the desired specific example and yield a numerical answer of arbitrary precision (just by adding the convenient number of terms of the corresponding series).

Disappointing as the mentioned—quite well known—ambiguities may be (specialists in the field are quite used to them by now), even more so is the fact that no general pattern seems to arise from our general formulae which might hint towards the physical understanding of the final *sign* of the energy. Here we have been able to demonstrate, without reasonable doubt, the existence of the two classes of Casimir force, attractive and repulsive, but are unable to give the rule for which one will show up at a particular instance. Further insight will be needed to clarify this point.

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